

Interactions of several replicas in the random field Ising model

 E. Brézin^{1,a} and C. De Dominicis²
¹ Laboratoire de Physique Théorique^b, École Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 05, France

² Service de Physique Théorique, CE Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 28 July 2000

Abstract. The replicated field theory of the random field Ising model involves the couplings of replicas of different indices. The resulting correlation functions involve a superposition of different types of long distance behaviours. However the $n = 0$ limit allows one to discuss the renormalization group properties in spite of this phenomenon. The attraction of pairs of replicas is enhanced under renormalization flow and no stable fixed point is found. Consequently, an instability occurs in the paramagnetic region, before one reaches the Curie line, signalling the onset of replica symmetry breaking.

PACS. 75.10.Nr Spin-glass and other random models – 05.50.+q Lattice theory and statistics (Ising, Potts, etc.)

1 Introduction

The simplest disordered system, the random field Ising model, remains 25 years after its introduction [1, 2], still far from understood. The initial ($d \rightarrow d - 2$) correspondence, between the RFIM in dimension d and the pure system in dimension $(d - 2)$ ([2–4]), was found to be certainly wrong in three dimensions, when it was proven that the RFIM possesses a non-zero critical temperature in three dimensions [5, 6]. It was then largely believed that non-perturbative effects spoiled the ($d \rightarrow d - 2$) correspondence, although it might be true to all orders in an expansion in powers of $\epsilon = 6 - d$. However it seems that the problem is not so much the occurrence of non-perturbative effects, but simply that the physics of the model did not reduce to a paramagnetic-ferromagnetic transition. Numerical experiments in the supposedly paramagnetic phase, pointing at large time scales before one reached the ordering temperature, as well as the theoretical approach of Mézard and Young [7], led to the idea that the phase diagram itself had to be reconsidered. Lowering the temperature in the disordered phase, it seems that one meets a glassy phase, characterized by replica symmetry breaking (RSB), in a region in which there is no magnetic ordering. In a recent article we have briefly examined the field theory associated with the random field Ising model [8], within the replica approach, as well as within a Langevin dynamics [9], which revealed a correspondence between

singularities in the zero-replica limit and long-time memory effects.

In the present article we return to the replica approach and discuss more precisely its renormalization group properties. We then discuss the instability related to an attractive interaction between two different replicas.

Let us first summarize the features of the replicated field theory of the RFIM. It is a ϕ^4 field theory, but with the following modifications:

(i) there are n fields ϕ_a , $a = 1, \dots, n$ and the only symmetry is the permutation of the replicas. This allows for five couplings namely $u_1 \sum_a \phi_a^4$, $u_2 \sum_{ab} \phi_a^3 \phi_b$, $u_3 \sum_{ab} \phi_a^2 \phi_b^2$,

$$u_4 \sum_{abc} \phi_a^2 \phi_b \phi_c, u_5 \left(\sum_1^n \phi_a \right)^4;$$

(ii) in the $n = 0$ limit the (bare) propagator at T_c is given by

$$G_{ab}(p) = \frac{\delta_{ab}}{p^2} + \frac{\Delta}{(p^2)^2}. \quad (1)$$

The second term results from the averaging over the random field $h(x)$:

$$\langle h(x)h(y) \rangle = \Delta \delta(x - y). \quad (2)$$

At first sight, the $1/p^4$ singularity of the propagator could imply that the upper critical dimension is eight, instead of six, but it will be argued that it is indeed six because of the $n = 0$ limit. Then the renormalization group studies which were conducted long ago [2–4] dealt only with the single coupling constant u_1 , and one Δ/p^4 propagator per

^a e-mail: brezin@physique.ens.fr

^b UMR 8549 CNRS

loop (contributions with less than one Δ/p^4 per loop being in any case infra-red subdominant). This limitation was the result of considering the averaging over the random field of connected correlation functions. However once one introduces the coupling constants u_2, \dots, u_5 , which involve several replicas, one must also consider contributions with more than one Δ/p^4 per loop. This modifies notably the conventional renormalization programme, and we shall carry it in some detail here in the zero replica limit. At the end we recover the $n = 0$ limit of the beta-functions found in [8] and thus confirm its main conclusion, namely the instability of the dimensional reduction fixed point.

Furthermore we are left with infra-red singular and attractive contributions to the 4-point function of type u_3 , *i.e.* corresponding to two distinct replicas. They can give rise to negative eigenvalues in the iteration of the related Bethe-Salpeter kernel, implying the occurrence of a glassy phase in the paramagnetic domain, before one reaches the Curie line [10]. We shall briefly return to this point at the end.

2 Upper critical dimension

Let us first write the Boltzmann weight for this replicated field theory:

$$\beta H = \int d^d x \left(\frac{1}{2} \sum_a [(\nabla \phi_a)^2 + r_o(\phi_a)^2] - \frac{\Delta}{2} \sum_{ab} \phi_a \phi_b + \frac{u_1}{4!} \sum_a \phi_a^4 + \frac{u_2}{3!} \sum_{ab} \phi_a^3 \phi_b + \frac{u_3}{8} \sum_{ab} \phi_a^2 \phi_b^2 + \frac{u_4}{4} \sum_{abc} \phi_a^2 \phi_b \phi_c + \frac{u_5}{4!} \left(\sum_a \phi_a \right)^4 \right). \quad (3)$$

Indeed in [8] it was argued that the standard derivation of a field theory from the spin model in the presence of a random field, namely the consideration of the fluctuations around mean field theory, did yield those five coupling constants. The corresponding propagator at the critical temperature is thus simply given by

$$G_{ab}(p) = \lim_{n \rightarrow 0} \left(\frac{\delta_{ab}}{p^2} + \frac{\Delta}{p^2(p^2 - n\Delta)} \right). \quad (4)$$

In view of this Δ/p^4 one does get infrared singularities in dimensions lower than eight. For instance there is an obvious one-loop contribution to the renormalization of u_3 proportional to $(\Delta u_1)^2$ which is singular at low external momenta as $1/p^{8-d}$. However if one considers higher loops they are either less singular or they vanish with n . Indeed the diagrams which would be maximally singular in eight dimensions contain a Δ/p^4 on every internal line. Therefore as soon as the diagram contains any vertex which is not connected to one or two of the four external lines, which is bound to happen at most at order five in perturbation theory, it contains at least one free sum over replica indices, and thus it vanishes with n .

Therefore the singularities that one encounters in these functions between eight and six dimensions are due to a finite number of graphs and thus there is no possibility of anomalous dimensions, which could modify the behavior given by simple dimensional analysis. This is very much like, say a six-point function at criticality, in a ϕ^4 theory: it is singular at low momentum for $d < 6$ but this singularity remains canonical down to four dimensions. Here similarly these singularities remain given by these few graphs down to $d = 6$, at which a full renormalization analysis becomes necessary: the upper critical dimension is six, not eight, because of the $n = 0$ limit.

3 Renormalization of the coupling constants

Given the effective Hamiltonian (3), we wish now to renormalize the four-point functions, corresponding to the five coupling constants u_j ($j = 1, \dots, 5$), retaining terms involving one Δ -propagator per loop or more. We consider the one-particle-irreducible four point function $\Gamma_{abcd}^{(4)}(p_1, \dots, p_4)$ and in order to minimize the number of momenta involved, we choose for simplicity the symmetric point

$$p_i \cdot p_j = \frac{p^2}{3} (5\delta_{ij} - 2) \quad (5)$$

compatible with momentum conservation $\sum_1^4 p_i = 0$. We work with a dimensionally regularized theory in dimension $d = 6 - \epsilon$, and will renormalize by minimal subtraction.

For simplicity let us first keep only u_1 and u_3 alone and examine what is happening. The four-point function of type 1, namely the one which involves a product of three Kronecker deltas $\delta_{ab}\delta_{ac}\delta_{ad}$, is given, at one-loop order, by

$$\Gamma_1^{(4)}(p) = u_1 - 3u_1 \frac{(\Delta u_1)}{\epsilon} \frac{1}{p^\epsilon} + \dots \quad (6)$$

If we multiply it by Δ , one sees that the real coupling constants which enter into all the diagrams which involve exactly one Δ per loop are

$$g_i = \Delta u_i. \quad (7)$$

Their (inverse length) dimension is $\epsilon = 6 - d$. Therefore in a minimal scheme, the relation between the bare g_1 and the dimensionless renormalized g_1^R is, at this order

$$g_1 = \mu^\epsilon g_1^R \left[1 + 3 \frac{g_1^R}{\epsilon} + O((g_1^R)^2) \right]. \quad (8)$$

Therefore the corresponding contribution to the beta function is

$$\beta_1 = -\epsilon g_1^R + 3(g_1^R)^2 + \dots \quad (9)$$

For the four-point function of type 3, namely the one which involves a product of two Kronecker deltas of the form $[\Gamma_3^{(4)}]_{ab} (\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd} + \delta_{ab}\delta_{cd})$, the situation is

different since we now meet two kinds of diagrams, those with one Δ per loop and those in which the number of Δ 's is equal to the number of loops plus one. On purely dimensional grounds we may decompose $\Delta\Gamma_3^{(4)}$ as

$$\Delta\Gamma_3^{(4)}(p) = \gamma_3(p) + \delta_3(p) \quad (10)$$

with

$$\begin{aligned} \gamma_3(p) &= g_3 - 2\frac{g_1 g_3}{\epsilon} \frac{1}{p^\epsilon} + \dots \\ \delta_3(p) &= \frac{\Delta}{p^{2+\epsilon}} g_1^2 + \dots \end{aligned} \quad (11)$$

(i) Consider first the most infra-red singular contribution $\delta_3(p)$. It comes from graphs with one Δ per loop, plus one. And its vertices can only be g_1 's, since the occurrence of one g_3 or more (like the insertion of more Δ 's) would entail some free replica summations vanishing with n . Furthermore the structure of those diagrams in $[\delta_3(p_1, p_2, p_3, p_4)]_{ab}$ is of the form

$$\begin{aligned} [\Gamma_3^{(4)}]_{ab}(p_1, \dots, p_4) &= (1 - \delta_{ab})\Delta^2 \\ &\times \int d^d q \Gamma_1^{(4)}(p_1, p_2, q, p_1 + p_2 - q) G_{ab}(q) G_{ab}(p_1 + p_2 - q) \\ &\times \Gamma_1^{(4)}(q, p_1 + p_2 - q, p_3, p_4) + \text{permutations}, \end{aligned} \quad (12)$$

in which $G_{ab}(q)$ is the renormalized propagator. The integral over q is convergent, since $(1 - \delta_{ab})$ selects the part of the propagator which falls off as $1/q^4$ (up to logarithms). Therefore the renormalization of $\Gamma_1^{(4)}$ and of the propagators is sufficient to make the diagrams contributing to $\delta_3(p)$ finite. No new renormalization is needed, and $\delta_3(p_1, p_2, p_3, p_4)$ satisfies a Callan-Symanzik equation *per se*.

(ii) $\gamma_3(p)$ comes from graphs with exactly one Δ per loop. They are all linear in g_3 since any higher power would again lead to free replica summations vanishing with n . Therefore the minimal renormalization of g_3 , defined as

$$g_3 = \mu^\epsilon g_3^R \left[1 + 2\frac{g_1^R}{\epsilon} + O((g_1^R)^2) \right]. \quad (13)$$

is sufficient to make $\gamma_3(p)$ finite. The beta-function for g_3 follows:

$$\beta_3 = -\epsilon g_3^R + 2g_1^R g_3^R + O(g_3^R (g_1^R)^2). \quad (14)$$

In general for the theory with the five coupling constants the same pattern governs the renormalization procedure. The functions $\Gamma_2^{(4)}$ and $\Gamma_3^{(4)}$ will both involve also terms with one Δ per loop, linear in g_2 and g_3 , which lead to a renormalization of g_2 and g_3 ; they have both contributions with one more Δ which are made finite by the previous renormalization of $\Gamma_1^{(4)}$. For $\Gamma_4^{(4)}$, one finds first terms linear in g_4 and quadratic in g_2 and g_3 which come from one Δ per loop; they lead to a renormalization of g_4 . Then one finds terms proportional to Δ/p^2 , linear in

g_2 and g_3 , and terms proportional to $(\Delta/p^2)^2$, which are made finite by the previous renormalization of $\Gamma_1^{(4)}$ and of the propagator. For $\Gamma_5^{(4)}$ the situation is again similar except that there are now terms up to $(\Delta/p^2)^3$ namely with k more Δ 's than the number of loops, $k = 0, \dots, 3$. The $k = 0$ terms lead to a renormalization of g_5 ; the other ones are finite as a consequence of previous renormalizations. The five beta functions at one-loop order are then [8]:

$$\begin{aligned} \beta_1 &= -\epsilon g_1 + 3g_1^2 \\ \beta_2 &= -\epsilon g_2 + 3g_1(g_2 + g_3) \\ \beta_3 &= -\epsilon g_3 + 2g_1(g_2 + g_3) \\ \beta_4 &= -\epsilon g_4 + 3g_1 g_4 + 4(g_2 + g_3)^2 \\ \beta_5 &= -\epsilon g_5 + 36g_4(g_2 + g_3). \end{aligned} \quad (15)$$

The dimensional reduction fixed point, namely $g_1 = \frac{1}{3}\epsilon + O(\epsilon^2)$ and $g_2 = \dots = g_5 = 0$ is unstable. (It is sufficient to notice that at this fixed point the matrix of derivatives $\frac{\partial\beta_i}{\partial g_j}$ has an eigenvalue equal to $\frac{\partial\beta_3}{\partial g_3} = -\frac{1}{3}\epsilon$.)

4 Field renormalization

At two-loop order a wave function renormalization appears. However *a priori* the random field introduces a privileged direction in the internal space along the unit vector $\mathbf{v} = \frac{1}{\sqrt{n}}(1, \dots, 1)$. We have thus to introduce the longitudinal and transverse components of the fields

$$\phi_L = \phi \cdot \mathbf{v} = \frac{1}{\sqrt{n}} \sum_{a=1}^n \phi_a \quad (16)$$

and

$$\phi_T = \phi - \phi_L \mathbf{v}. \quad (17)$$

A priori they are both renormalized by different factors and we define the renormalized fields through

$$\phi_L = \sqrt{Z_L} \Phi_L, \quad \phi_T = \sqrt{Z_T} \Phi_T, \quad (18)$$

in which the n fields Φ have finite correlation functions when $\epsilon = (6 - d)$ goes to zero. In terms of these fields the quadratic terms in the action have the form

$$\beta H_0 = \int d^d x \left(\frac{1}{2} [(\nabla \phi_L)^2 + (\nabla \phi_T)^2] - n \frac{\Delta}{2} \phi_L^2 \right), \quad (19)$$

i.e. in terms of the renormalized fields

$$\beta H_0 = \int d^d x \left(\frac{1}{2} [Z_L (\nabla \Phi_L)^2 + Z_T (\nabla \Phi_T)^2] - n Z_L \frac{\Delta}{2} \Phi_L^2 \right). \quad (20)$$

This gives for the renormalized two-point function

$$\Gamma_{ab}^{(2)}(p) = Z_L (p^2 - n\Delta) \frac{1}{n} + Z_T p^2 (\delta_{ab} - \frac{1}{n}) - \Sigma_{ab}, \quad (21)$$

in which we have used the projectors along the vector \mathbf{v} , the constant matrix whose elements are $1/n$, and the projector transverse to \mathbf{v} , the matrix $\delta_{ab} - 1/n$; Σ_{ab} contains all the self-energy diagrams.

From this expression one sees that it is expected that the wave function renormalization of the longitudinal part of the propagator Z_L renormalizes as well Δ , the variance of the random field. This is not obvious *a priori*; in the appendix an explicit calculation to two-loop order is given, which shows that to this order $Z_T = Z_L$ and that Δ does not acquire an independent renormalization. In a Langevin dynamics of the same problem [9] the non-renormalization of Δ is a natural consequence of the non-renormalization of the temperature.

5 Instability

A derivation of the Landau theory with the five ϕ^4 coupling constants, was given in [8]. It was shown there that the coupling constant $g_3 = u_3\Delta$, which couples two distinct replicas, was attractive. We note that under renormalization g_3 is pushed further away from the origin. It is thus natural to examine whether this attractive and singular interaction (10) may lead to an instability. This question has been considered in some details by several authors [10–13]. Here we wish to keep to very simple arguments.

Therefore we now consider whether the Bethe-Salpeter kernel, for a pair of replicas of different indices, might develop a vanishing eigenvalue, thereby signaling bound state formation. Note that couplings other than g_3 vanish with n under iteration in the channel of two (distinct) replicas. The simplest iterative kernel (besides g_3) is an attractive “bubble”, *i.e.* a δ_3 -like contribution followed by two propagators, *i.e.*

$$\tilde{g}_1^2 \frac{1}{[(p-q)^2]^2} \int d^d k \frac{1}{(k^2)^2 [(k-q)^2]^2}. \quad (22)$$

An instability takes place whenever the spectrum of the operator

$$\hat{h}\psi = p^4\psi(p) - \tilde{g}_1^2 \int d^d q B(p-q)\psi(q) \quad (23)$$

has a vanishing eigenvalue, in which

$$B(p) = \int d^d k \frac{1}{(k^2)^2 [(k-p)^2]^2}. \quad (24)$$

In position space this operator is

$$\hat{h} = [(\nabla)^2]^2 - C\tilde{g}_1^2 \frac{1}{r^{2d-8}} \quad (25)$$

in which C is a positive constant. The four derivatives may balance the singularity at the origin of the attractive potential whenever $2d - 8 \leq 4$. In that range the spectrum consists of bound states and positive energy scattering states. A simple scaling shows that the binding energies,

in the domain $d \leq 6$ are proportional to $|\tilde{g}_1|^{4/(6-d)}$. Therefore whenever $|\tilde{g}_1|^{4/(6-d)}$ is larger than some critical value a bound state at zero energy appears, signaling an instability. Since \tilde{g}_1 is proportional to Δ , it means that, for Δ small, there is a small domain of size proportional to $\Delta^{4/(6-d)}$ above the Curie line, in which the attraction between pairs of distinct replicas generate an instability.

Quite generally it can be shown that the \hat{h} operator of (23) is, beyond the one-loop approximation, the Jacobian matrix of the Legendre transform [13] with respect to a source $\Delta_{ab}(p)$

$$W(\Delta_{ab}) + \Gamma(G_{ab}) = \frac{1}{2} \sum_p \sum_{ab} \Delta_{ab}(p) G_{ab}(p) \quad (26)$$

i.e.

$$\begin{aligned} \hat{h}_{ab}(p; p') &= \frac{\partial^2 \Gamma}{\partial G_{ab}(p) \partial G_{ab}(p')} \\ &= [G_{aa}(p)]^{-1} [G_{bb}(p)]^{-1} \delta_{p+p'; 0} \\ &\quad - \Gamma_3^{(4)}(p, -p; p', -p'). \end{aligned} \quad (27)$$

On the other hand that double derivative has been identified with the replicon component of the Hessian matrix [10, 13], around the replica symmetric solution for the connected ($G_{aa} = G$) and disconnected (or connected through the random averaging, $G_{ab} = \tilde{G}$) components. Thus the instability, *i.e.* negative eigenvalues of \hat{h} , will force the emergence of a replica symmetry broken solution [10, 11].

To conclude in one sentence, it may be said that the breakdown of dimensional reduction is not simply that the ϵ -expansion needs to be corrected by non-analytic terms, but follows from a change in the phase diagram itself with the emergence of a new, glassy, phase.

Appendix: The propagator at two-loop order

The contributions to the self-energy due to the coupling constant g_1 vanish at one-loop in the minimal subtraction scheme. At two-loop the singular terms of the self-energy are given by

$$\Sigma_{ab} = \tilde{g}_1^2 [A(p)\delta_{ab} + \Delta B(p)] \quad (A.1)$$

with

$$A(p) = \frac{1}{2} \int d^d q_1 d^d q_2 \frac{1}{q_1^4} \frac{1}{q_2^4} \frac{1}{(p+q_1+q_2)^2} \quad (A.2)$$

and

$$B(p) = \frac{1}{6} \int d^d q_1 d^d q_2 \frac{1}{q_1^4} \frac{1}{q_2^4} \frac{1}{(p+q_1+q_2)^4}, \quad (A.3)$$

$$\tilde{g}_1 = g_1 \frac{2\pi^{d/2}}{(2\pi)^d \Gamma(d/2)}. \quad (A.4)$$

A lengthy, but standard, dimensional calculation in which $d = 6 - \epsilon$, leads to

$$\begin{aligned}
 A &= -\frac{p^{2+2\epsilon} (1 - \epsilon/2)^2 (1 - \epsilon/4)^2}{12\epsilon (1 - 3\epsilon/2)(1 - 3\epsilon/4)} \\
 &\quad \times \frac{\Gamma^5(1 - \epsilon/2)\Gamma(1 + \epsilon)}{\Gamma(1 - 3\epsilon/2)} \\
 &= -\frac{p^2}{12\epsilon} + \text{finite}
 \end{aligned} \tag{A.5}$$

and

$$\begin{aligned}
 B &= \frac{p^{-2\epsilon} (1 - \epsilon/2)^2 (1 - \epsilon/4)^2 (1 + \epsilon)}{12\epsilon (1 + \epsilon/2)(1 + \epsilon/4)(1 - \epsilon)} \\
 &\quad \times \frac{\Gamma^5(1 - \epsilon/2)\Gamma(1 + \epsilon)}{\Gamma(1 + \epsilon/2)\Gamma(1 - \epsilon)} \\
 &= \frac{1}{12\epsilon} + \text{finite}.
 \end{aligned} \tag{A.6}$$

At this order the one-particle irreducible two-point function is given by

$$\Gamma_{ab}^{(2)}(p) = Z_T(\delta_{ab} - \frac{1}{n})p^2 + Z_L \frac{1}{n}(p^2 - n\Delta) - \Sigma_{ab} \tag{A.7}$$

and thus in the minimal subtraction scheme, we find that,

since A and B diverge with the same $\pm \frac{1}{12\epsilon}$

$$Z_T = Z_L = 1 + \frac{\tilde{g}_1^2}{12\epsilon} + O(\tilde{g}_1^3), \tag{A.8}$$

and that indeed Δ does not require any additional renormalization.

References

1. Y. Imry, S.K. Ma, Phys. Rev. Lett. **35**, 1399 (1975).
2. A. Aharony, Y. Imry, S.K. Ma, Phys. Rev. Lett. **37**, 1364 (1976).
3. A.P. Young, J. Phys C **10**, L257 (1977).
4. G. Parisi, N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979).
5. J.Z. Imbrie, Phys. Rev. Lett. **43**, 1747 (1984).
6. J. Bricmont, A. Kupiainen, Phys. Rev. Lett. **59**, 1829 (1987).
7. M. Mézard, A.P. Young., Europhys. Lett. **18**, 653 (1992).
8. E. Brézin, C. De Dominicis, Europhys. Lett. **44**, 13 (1998).
9. E. Brézin, C. De Dominicis, C.R. Acad. Sci. **327**, 383 (1999).
10. M. Mézard, P. Young, Europhys. Lett. **18**, 653 (1992).
11. J. de Almeida, R. Bruinsma, Phys. Rev. B **35**, 7267 (1987).
12. M. Mézard, R. Monasson, Phys. Rev. B **50**, 7199 (1994).
13. C. De Dominicis, H. Orland, T. Temesvari, J. Phys. I France **5**, 987 (1995).